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Homogeneous Equations:—

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$. Where $f(x,y)$ and $\phi(x,y)$ are homogeneous function of x, y and of the same degree is said to be homogeneous.

Working Rule:— Every homogeneous equation of the above type can be solved by putting $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

After making the substitution for y and $\frac{dy}{dx}$, the given differential equation can be reduced to the form in which the variables are separable. Then after performing the integration we replace y by $\frac{v}{x}$ and the required result follows.

Ex-1. Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$

Solution:— The given equation can be Written as

$$(x^3+y^3)\,dy = x^2y\,dx.$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \text{—(1)}$$ Put $y = vx$ so that

$$\frac{dy}{dx} = v.1 + x\frac{dv}{dx}.$$

Hence the equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2vx}{x^3+v^3x^3} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3} = -\frac{v^4}{1+v^3}$$

$$\Rightarrow \frac{dx}{x} = -\frac{1+v^3}{v^4}dv.$$

Integrating, we get

$$\int \frac{dx}{x} = -\int \frac{1+v^3}{v^4}dv$$

$$\Rightarrow \log x = -\int\left(\frac{1}{v^4} + \frac{1}{v}\right)dv$$

$$= -\int\left(v^{-4} + \frac{1}{v}\right)dv = \left[\frac{1}{-3v^3} + \log v\right] + c$$

$$= \frac{1}{3v^3} - \log v + c = \frac{x^3}{3y^3} - \log\left(\frac{y}{x}\right) + c$$

$$= \frac{x^3}{3y^3} - \log y + \log x + c$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} + c$$

Which is the required solution.

Ex-2. Solve $x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$

Solution:- from the given equation, we have

$$x\,dy = \left(y + \sqrt{x^2+y^2}\right)dx.$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \longrightarrow (1)$$

Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence the given equation reduces to

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x}$$

$$= \frac{vx + x\sqrt{1+v^2}}{x} = v + \sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

Integrating, we have

$$\int dx/x = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow \log x = \log(v + \sqrt{1+v^2}) + \log c$$

$$= \log c(v + \sqrt{1+v^2})$$

$$\Rightarrow x = c(v + \sqrt{1+v^2})$$

$$= c\left(\frac{y}{x} + \sqrt{1+y^2/x^2}\right) \quad (\text{From } v = y/x)$$

$$= c\,\frac{y + \sqrt{x^2+y^2}}{x}$$

$$\therefore x^2 = c(y + \sqrt{x^2+y^2})$$

Which is the required solution.

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